

[Note]

Relⁿ betⁿ $S_{y_3}(f)$, $M(f)$, $T(f)$.

$$\text{is } S_{y_3}(f) = \frac{2 \cdot M(f) + 1 \cdot T(f)}{2+1} \quad \text{--- (9)}$$

eg find $\int_0^1 e^{-x^2} dx$ by Trapezoidal, Simpson y_3 and Simpson $\frac{3}{8}$ rule.

$$\text{by } T(f) = \frac{1-0}{2} (e^{-0^2} + e^{-1^2})$$

$$= \frac{1}{2} (1 + \frac{1}{e}) = 0.68394$$

$$S_{y_3}(f) = \frac{1-0}{6} (e^{-0^2} + 4e^{-(\frac{1}{2})^2} + e^{-1^2})$$

$$= \frac{1}{6} (1 + 4e^{-\frac{1}{4}} + e^{-1})$$

$$= 0.74718$$

$$S_{3/8}(f) = \frac{3(1-0)}{8 \times 3} (e^{-0^2} + 3\{e^{-(\frac{1}{3})^2} + e^{-(\frac{2}{3})^2}\} + e^{-1^2})$$

$$= \frac{1}{8} (1 + 3(e^{-\frac{1}{9}} + e^{-\frac{4}{9}}) + e^{-1})$$

$$= 0.74699$$

Degree of precision :-

Let $R_{n+1}(f)$ be a $n+1$ point rule given by

$$R_{n+1}(f) = w_0 f_0 + w_1 f_1 + \dots + w_n f_n \quad \text{--- (1)}$$

$$\Rightarrow I(f) = I(f) + E(f) \quad \text{--- (2)}$$

(1) The rule $R_{n+1}(f)$ is said to be exact

if $E(f) = 0$ in (2).

that means for the exactness we consider the monomial x^m (where m is non negative integer) such that

$$I(x^m) = R_{n+1}(x^m) + E(x^m) \quad \text{where } E(x^m) = 0.$$

$$\Rightarrow I(x^m) = R_{n+1}(x^m)$$

(3) The degree of precision of the rule $R_n(f)$ is denoted as d (re integer) if the rule is exact for all monomials of degree $\leq d$, but fails to be exact for monomial of degree $d+1$.

i.e d is the degree of precision of rule $R_{n+1}(f)$ if

$$E(x^m) = \begin{cases} 0 & 0 \leq m \leq d \\ \text{non zero} & m = d+1 \end{cases}$$

eg on Trapezoidal Rule,

$$1. I(f) = R_2(f) + E(f) \\ = R_2(f) + \frac{1}{12}(b-a)^3 f''(\eta)$$

The Error term will be zero if we take $f(x) = 1$ or x . (i.e $m=0,1$)

$$\text{Since } f''(1) = 0 \quad f''(x) = 0$$

$$\text{But for } f(x) = x^2, \quad f''(x^2) \neq 0$$

So the degree of precision is 1 for T.R.

2. The degree of precision of both Simpson's $\frac{1}{3}$ and $\frac{3}{8}$ Rule is 3 Since their error terms involves 4th derivative of the integrand.

Compound quadrature rule (Composite rule)
 For large interval of integration we require
 compound quadrature Rule

Compound Rectangular Rule:-

Let the points are $x_0=a, x_1, x_2, \dots, x_n=b$.

i.e. $[a, b]$ divided into
 n subintervals of equal
 length h .



$$h = \frac{b-a}{n}$$

$$x_i = x_0 + i h, \quad i = 0(1)n$$

we know $I(f) = \int_a^b f(x) dx$

Let us consider the i th subinterval $[x_{i-1}, x_i]$

so integration in this region is

$$I_i = \int_{x_{i-1}}^{x_i} f(x) dx \approx (x_i - x_{i-1}) f(x_{i-1})$$

(using rectangular rule)

$$\text{so } I(f) = \sum_{i=1}^n \int_{x_{i-1}}^{x_i} f(x) dx$$

$$I(f) \approx \sum_{i=1}^n \left\{ (x_i - x_{i-1}) f(x_{i-1}) + \frac{1}{2} (x_i - x_{i-1})^2 f'(\eta_i) \right\}$$

where $\eta_i \in (x_{i-1}, x_i)$

Leaving the error term

$$I_{CLR}(f) \approx \sum_{i=1}^n (x_i - x_{i-1}) f(x_{i-1})$$

$$\approx (x_1 - x_0) f(x_0) + (x_2 - x_1) f(x_1) + \dots + (x_n - x_{n-1}) f(x_{n-1})$$

$$I_{CLR}(f) \approx h [f(x_0) + f(x_1) + \dots + f(x_{n-1})] \quad (4)$$

This is called compounded left rectangular rule. And the error term is given by

$$E_{CLR}(f) = \sum_{i=1}^n \frac{1}{2} (x_i - x_{i-1})^2 f'(x_i)$$

$$= \frac{1}{2} [(x_1 - x_0)^2 f'(x_1) + (x_2 - x_1)^2 f'(x_2) + \dots + (x_n - x_{n-1})^2 f'(x_n)]$$

$$= \frac{h^2}{2} [f'(x_1) + f'(x_2) + \dots + f'(x_n)]$$

$$\boxed{E_{CLR}(f) = n \frac{h^2}{2} f'(y)} \quad \text{where } y \approx x_1, x_2, \dots, x_n, \quad a \leq y \leq b \quad (11)$$

$$\Rightarrow E_{CLR}(f) = n \frac{(b-a)^2}{2n^2} f'(y) \quad (\because h = \frac{b-a}{n})$$

$$\Rightarrow \boxed{E_{CLR}(f) = \frac{(b-a)^2}{2n} f'(y)} \quad (12)$$

eg/ Evaluate $\int_0^1 x \, dx$ by using CLR rule, choosing

$$n = 2, 3, 4, 5, 6$$

Ans- The given interval is $[0, 1]$ which is divided into n equal subintervals

$$f(x) = x$$

$$R_{CLR}(f) = h [f(x_0) + f(x_1) + \dots + f(x_n)]$$

$$= \frac{1}{n} \left[0 + \frac{1}{n} + \frac{2}{n} + \dots + \frac{n-1}{n} \right]$$

$$= \frac{1}{n^2} [1 + 2 + \dots + (n-1)]$$



$$h = \frac{1-0}{n} = \frac{1}{n}$$

$$\begin{aligned} x_0 &= x_0 + 0h \\ x_1 &= 0 + \frac{1}{n} \\ x_2 &= 0 + \frac{2}{n} \end{aligned}$$

$$= \frac{1}{n^2} \frac{n(n-1)}{2}$$

$$= \frac{1}{2} \left(\frac{n-1}{n} \right)$$

$$= \frac{1}{2} \left(1 - \frac{1}{n} \right)$$

$$\left[\because 1+2+\dots+n = \frac{n(n+1)}{2} \right]$$

So	n	$P(f) = \frac{1}{2} \left(1 - \frac{1}{n} \right)$
	2	$\frac{1}{2} \left(1 - \frac{1}{2} \right) = \frac{1}{4} = 0.25$
	3	$\frac{1}{2} \left(1 - \frac{1}{3} \right) = \frac{1}{3} = 0.\bar{3}$
	4	$\frac{1}{2} \left(1 - \frac{1}{4} \right) = \frac{3}{8} = 0.375$
	5	$\frac{1}{2} \left(1 - \frac{1}{5} \right) = \frac{2}{5} = 0.4$
	6	$\frac{1}{2} \left(1 - \frac{1}{6} \right) = \frac{5}{12} = 0.41666$

Exact value of $P(f) = \int_0^1 f(x) dx = \left[\frac{x^2}{2} \right]_0^1$

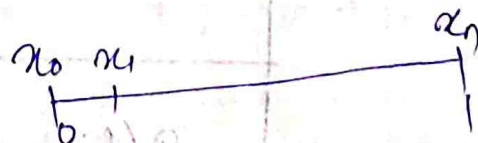
$$= \frac{1}{2} = 0.5$$

eg/ Evaluate integral $\int_0^1 (1+x) dx$ by CLR rule
for $n = 2, 4, 6, 8, 10$.

Ans/- $h = \frac{1-0}{n} = \frac{1}{n}$

$$x_0 = 0, \quad x_n = 1$$

$$x_i = x_0 + i h$$



and $f(x) = 1+x$

$$\text{So } R_{CLR}(f) = h \left[f(x_0) + f(x_1) + \dots + f(x_{n-1}) \right]$$

$$= \frac{1}{n} \left[1 + \left(1 + \frac{1}{n} \right) + \left(1 + \frac{2}{n} \right) + \dots + \left(1 + \frac{n-1}{n} \right) \right]$$